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The gap of the eigenvalues for p-forms and harmonic p-forms of constant length

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Abstract

We study the kth positive eigenvalue $\lambda_k^{(p)}(M, g)$ of the Laplacian on p-forms for a connected oriented closed Riemannian manifold (M, g). If all non-trivial harmonic p-forms on (M, g) have constant length, then it follows that $\lambda_k^{(p)}(M, g) \leq \lambda_k^{(0)}(M, g)$ for all $k \geq 1$.

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1. Introduction

Let (M^m, g) be a connected oriented closed Riemannian manifold of dimension m. We denote by $\lambda_k^{(p)}(M, g)$ the kth positive eigenvalue of the Laplacian $\Delta = d\delta + \delta d$ acting on p-forms on (M, g) counted with multiplicity.

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In our previous papers [14,16], we studied the gap

$$\mathrm{Gap}^{(p,0)}(M,g) := \lambda_1^{(p)}(M,g) - \lambda_1^{(0)}(M,g)$$

of the first eigenvalues of (M, g). We proved that for any $m \geq 4$ dimensional connected oriented closed manifold M and any $2 \leq p \leq m-2$, there exist three metrics $g_{p,i}$ $(i = 1, 2, 3)$ on M such that $\mathrm{Gap}^{(p,0)}(M, g_{p,i})$ is positive, negative and zero (Theorem 1.2 in [16]). In the case of $p = 1$, since Δ and d commute, it always holds $\lambda_1^{(1)} \leq \lambda_1^{(0)}$, that is, $\mathrm{Gap}^{(1,0)} \leq 0$. We also obtained a similar result. For any $m \geq 3$ dimensional connected oriented closed manifold M, there exist two metrics g_i $(i = 1, 2)$ on M such that $\mathrm{Gap}^{(1,0)}$ is negative and zero (Theorem 1.1 in [16]). Thus, we see that the sign of the gap $\mathrm{Gap}^{(p,0)}$ is not a topological invariant of closed manifolds.

On the other hand, there are some relations between the sign of the gap and the geometry of (M, g). For example, if $\mathrm{Gap}^{(1,0)}$ is negative for a connected closed Einstein manifold with positive Ricci curvature, then the identity map is weakly stable as a harmonic map (Theorem 1.3 in [14]). If a closed Riemannian manifold has a non-trivial parallel p-form, then $\mathrm{Gap}^{(p,0)}$ is non-positive (Proposition 1.3 in [16]).

In the present paper, we study an extension of this result. We prove that if all non-trivial harmonic p-forms on (M, g) have constant length, then $\mathrm{Gap}^{(p,0)} \leq 0$. More precisely, we have the following.

Theorem 1.1. *Let (M, g) be a connected oriented closed Riemannian manifold with the pth Betti number $b_p(M) \geq 1$. If all harmonic p-forms have constant length, then for all $k \geq 1$ it follows that*

$$\lambda_k^{(p)}(M,g) \leq \lambda_k^{(0)}(M,g).$$

Furthermore, the equality $\lambda_1^{(p)}(M,g) = \lambda_1^{(0)}(M,g)$ holds if and only if $f\varphi$ is a first positive eigen p-form on (M, g), where f is a first positive eigenfunction and φ is a non-trivial harmonic p-form of constant length.

Although the length of parallel forms is constant, note that Theorem 1.1 does not contain Proposition 1.3 in [16]. We here need the condition that all harmonic p-forms have constant length. So, we study the case where a one non-trivial harmonic p-form has constant length.

Proposition 1.2. *Let (M, g) be a connected oriented closed Riemannian manifold with the pth Betti number $b_p = b_p(M) \geq 1$. If there exists a non-trivial harmonic p-forms of constant length, then for all $k \geq 1$ it follows that*

$$\lambda_k^{(p)}(M,g) \leq \lambda_{b_p+k-1}^{(0)}(M,g).$$

There exist closed Riemannian manifolds such that all harmonic p-forms are of constatnt length and not parallel. We exhibit such an example in Section 3, Example 3.3. We also give examples such that $\mathrm{Gap}^{(p,0)}$ is zero and negative in Section 3.

A typical example of Theorem 1.1 is a connected oriented closed homogeneous Riemannian manifold (cf. [4]). Another example is an oriented closed geometrically formal Riemannian manifold, that is, exterior products of harmonic forms are still harmonic. Then, the length of all harmonic forms on an oriented closed geometrically formal Riemannian manifold is constant. For the details, see [9,10,13]. Recently, Guerini and Savo [6,7] study the gap, $\mathrm{Gap}^{(p,p-1)} = \lambda_1^{(p)} - \lambda_1^{(p-1)}$, for compact Riemannian manifolds with boundary.

By the contraposition to Theorem 1.1, we have the following.

Corollary 1.3. *Let (M, g) be a connected oriented closed Riemannian manifold with $b_p(M) \geq 1$. If $\lambda_1^{(p)}(M, g) > \lambda_1^{(0)}(M, g)$, then there exists a harmonic p-form whose length is not constant on M. In particular, (M, g) is neither homogeneous nor geometrically formal.*

Next, we consider an estimate of the eigenvalue $\lambda_k^{(p)}$ in terms of geometrical data of Riemannian manifolds. While $\lambda_k^{(0)}$ is estimated above in terms of the lower bounds of the Ricci curvature and the diameter by Cheng [2], $\lambda_k^{(p)}$ can not be estimated in terms of these geometrical data (see Example 5.2 in [1]). However, if the length of all non-trivial harmonic p-forms is constant, by combining Theorem 1.1 or Proposition 1.2 with the estimate for $\lambda_k^{(0)}$ by Cheng, we obtain estimates of $\lambda_k^{(p)}$.

Corollary 1.4. *Let (M, g) be a connected oriented closed Riemannian manifold with $b_p(M) = b_p \geq 1$, the Ricci curvature $Ric \geq -\kappa^2$ and the diameter diam $(M,g) \geq d$, where κ, d are positive constants.*

(1) *If all harmonic p-forms have constant length, then it holds that for any $k \geq 1$*

$$\lambda_k^{(p)}(M, g) \leq \frac{m-1}{4}\kappa^2 + \frac{c(m)}{d^2}k^2.$$

(2) *If there exists a non-trivial harmonic p-form of constant length, then it holds that for any $k \geq 1$*

$$\lambda_k^{(p)}(M, g) \leq \frac{m-1}{4}\kappa^2 + \frac{c(m)}{d^2}(b_p + k - 1)^2.$$

The constant $c(m) > 0$ depends only on the dimension m.

2. Proofs

To prove Theorem 1.1 and Proposition 1.2, we use the following lemma.

Lemma 2.1. *Let (M, g) be an oriented Riemannian manifold. For 1-forms θ_1, θ_2 and p-form φ on M, it follows that*

$$\langle \theta_1 \wedge \varphi, \theta_2 \wedge \varphi \rangle + \langle \theta_1 \wedge *\varphi, \theta_2 \wedge *\varphi \rangle = \langle \theta_1, \theta_2 \rangle \, |\varphi|^2, \tag{2.1}$$

where $*$ is the Hodge star operator.

Proof. For 1-form θ and p-form φ, we have only to prove that

$$|\theta \wedge \varphi|^2 + |\theta \wedge *\varphi|^2 = |\theta|^2\,|\varphi|^2. \tag{2.2}$$

In fact, if we set $\theta = \theta_1 + \theta_2$ for (2.2), then it is easy to obtain (2.1).

Now, we show the equality (2.2). Since $\langle \theta \wedge \varphi, \psi \rangle = \langle \varphi, i_{\theta^\sharp}\psi \rangle$ for any $(p+1)$-form ψ, we have

$$|\theta \wedge \varphi|^2 = \langle \varphi, i_{\theta^\sharp}(\theta \wedge \varphi)\rangle = \langle \varphi, i_{\theta^\sharp}(\theta)\varphi - \theta \wedge i_{\theta^\sharp}(\varphi)\rangle = |\theta|^2|\varphi|^2 - \langle \varphi, \theta \wedge i_{\theta^\sharp}(\varphi)\rangle. \tag{2.3}$$

Similarly, since $*$ is isometry, we have

$$|\theta \wedge *\varphi|^2 = |\theta|^2|\varphi|^2 - \langle *\varphi, \theta \wedge i_{\theta^\sharp}(*\varphi)\rangle.$$

If we denote by v_g the volume form of (M, g), then we have

$$\begin{aligned}\langle *\varphi, \theta \wedge i_{\theta^\sharp}(*\varphi)\rangle v_g &= \varphi \wedge \theta \wedge i_{\theta^\sharp}(*\varphi) = (-1)^p i_{\theta^\sharp}(\varphi) \wedge \theta \wedge *\varphi + i_{\theta^\sharp}(\theta)\varphi \wedge *\varphi \\ &= |\theta|^2|\varphi|^2 v_g - (-1)^{p(m-p)} * \varphi \wedge \theta \wedge i_{\theta^\sharp}(\varphi) \\ &= \{|\theta|^2|\varphi|^2 - \langle \varphi, \theta \wedge i_{\theta^\sharp}(\varphi)\rangle\} v_g.\end{aligned}$$

Hence,

$$|\theta \wedge *\varphi|^2 = \langle \varphi, \theta \wedge i_{\theta^\sharp}(\varphi)\rangle. \tag{2.4}$$

Thus, from (2.3) and (2.4), we obtain the equality (2.2). □

Now we prove Theorem 1.1.

Proof. From the assumption, we can take harmonic p-forms $\{\varphi_1, \ldots, \varphi_{b_p}\}$ $(b_p \geq 1)$ such that

$$\langle \varphi_i, \varphi_j \rangle = \delta_{ij}. \tag{2.5}$$

In fact, if $\{\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{b_p}\}$ is an orthonormal basis of the space of harmonic p-forms with respect to the L^2-inner product on (M, g), then it is enough to set $\varphi_i := \sqrt{\mathrm{vol}(M, g)}\tilde{\varphi}_i$, where $\mathrm{vol}(M, g)$ means the volume of (M, g). Since the length of all harmonic p-forms is constant,

$$2\langle \varphi_i, \varphi_j \rangle = |\varphi_i + \varphi_j|^2 - |\varphi_i|^2 - |\varphi_j|^2$$

is constant on M. Hence, we have

$$\langle \varphi_i, \varphi_j \rangle = \frac{1}{\mathrm{vol}(M, g)}(\varphi_i, \varphi_j)_{L^2} = (\tilde{\varphi}_i, \tilde{\varphi}_j)_{L^2} = \delta_{ij}.$$

Let f_i $(i = 1, \ldots, k)$ be an ith orthonormal eigenfunction on (M, g). Note that

$$(df_i, df_j)_{L^2} = \lambda_i^{(0)}(M, g)\delta_{ij} \quad \text{and} \quad \int_M f_i\, \mathrm{d}\mu_g = 0. \tag{2.6}$$

We set the p-form

$$\omega_i := f_i\varphi_1 \quad (i = 1, 2, \ldots).$$

From (2.6), it follows that

$$\begin{aligned}(\omega_i, \varphi_j)_{L^2} &= \textstyle\int_M f_i\langle\varphi_1, \varphi_j\rangle \,\mathrm{d}\mu_g = \delta_{1j}\int_M f_i \,\mathrm{d}\mu_g = 0,\\ (\omega_i, \omega_j)_{L^2} &= \textstyle\int_M f_i f_j\langle\varphi_1, \varphi_1\rangle \,\mathrm{d}\mu_g = (f_i, f_j)_{L^2} = \delta_{ij}.\end{aligned} \tag{2.7}$$

If we take the linear subspace $E := \langle\varphi_1, \ldots, \varphi_{b_p}, \omega_1, \ldots, \omega_k\rangle_{\mathbb{R}}$ of the space of smooth p-forms on M, by (2.7), then $\dim E = k + b_p$. By the min–max principle, we have

$$\lambda_k^{(p)}(M, g) \le \sup_{\omega\neq 0\in E}\left\{\frac{\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2}{\|\omega\|_{L^2}^2}\right\}. \tag{2.8}$$

We have only to estimate the right hand side of (2.8) from above. For any non-zero element $\omega \in E$, we may write $\omega = \sum_{i=1}^{b_p} a_i\varphi_i + \sum_{j=1}^{k} c_j\omega_j \in E$, where $a_i, c_j \in \mathbb{R}$ and one of them is non-zero. By (2.6) and (2.7), the denominator of (2.8) is

$$\|\omega\|_{L^2}^2 = \left\|\left(a_1 + \sum_{j=1}^{k} c_j f_j\right)\varphi_1\right\|_{L^2}^2 + \sum_{i=2}^{b_p} a_i^2\|\varphi_i\|_{L^2}^2 = \sum_{i=1}^{b_p} a_i^2 \mathrm{vol}(M, g) + \sum_{j=1}^{k} c_j^2. \tag{2.9}$$

Next, we compute the numerator of (2.8). Since φ_j is harmonic, we have

$$\begin{aligned}\|d\omega\|_{L^2}^2 &= \left\|\sum_{j=1}^{k} c_j d(f_j\varphi_1)\right\|_{L^2}^2 = \left\|\sum_{j=1}^{k} c_j df_j \wedge \varphi_1\right\|_{L^2}^2\\ &= \sum_{i,j=1}^{k} c_i c_j (df_i \wedge \varphi_1, df_j \wedge \varphi_1)_{L^2}.\end{aligned} \tag{2.10}$$

Similarly, since $\delta = (-1)^{mp+m+1} * d*$ and $*$ is isometry, we have

$$\begin{aligned}\|\delta\omega\|_{L^2}^2 &= \left\|\sum_{j=1}^{k} c_j d * (f_j\varphi_1)\right\|_{L^2}^2 = \left\|\sum_{j=1}^{k} c_j df_j \wedge *\varphi_1\right\|_{L^2}^2\\ &= \sum_{i,j=1}^{k} c_i c_j (df_i \wedge *\varphi_1, df_j \wedge *\varphi_1)_{L^2}.\end{aligned} \tag{2.11}$$

Hence, from (2.10), (2.11) and Lemma 2.1, the numerator is

$$\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 = \sum_{i,j=1}^{k} c_i c_j \int_M \{\langle df_i \wedge \varphi_1, df_j \wedge \varphi_1\rangle + \langle df_i \wedge *\varphi_1, df_j \wedge *\varphi_1\rangle\} \,\mathrm{d}\mu_g$$

$$= \sum_{i,j=1}^{k} c_i c_j \int_M \langle df_i, df_j \rangle \, |\varphi_1|^2 \, \mathrm{d}\mu_g = \sum_{i,j=1}^{k} c_i c_j (df_i, df_j)_{L^2}$$

$$= \sum_{i,j=1}^{k} c_i c_j \lambda_i^{(0)}(M, g)\delta_{ij} \le \lambda_k^{(0)}(M, g) \sum_{j=1}^{k} c_j^2. \tag{2.12}$$

Therefore, by substituting (2.9), (2.12) to (2.8), we obtain $\lambda_k^{(p)}(M, g) \le \lambda_k^{(0)}(M, g)$.

Next, we consider the condition that the equality $\lambda_1^{(p)}(M, g) = \lambda_1^{(0)}(M, g)$ holds. Let f and φ be a first positive eigenfunction and a harmonic p-form of constant length on (M, g), respectively. If $f\varphi$ is a first positive eigen p-form on (M, g), by Lemma 2.1, then we have

$$\lambda_1^{(p)}(M, g) = \frac{\|d(f\varphi)\|_{L^2}^2 + \|d(f * \varphi)\|_{L^2}^2}{\|f\varphi\|_{L^2}^2} = \frac{\|df\|_{L^2}^2}{\|f\|_{L^2}^2} = \lambda_1^{(0)}(M, g).$$

We prove the opposite direction. Suppose that $\lambda_1^{(p)}(M, g) = \lambda_1^{(0)}(M, g)$. Since $f\varphi$ is orthogonal to the space of harmonic p-forms $\mathbf{H}^p(M, g)$ on (M, g) from (2.7), by the min–max principle, we have

$$\lambda_1^{(p)}(M, g) = \inf \left\{ \frac{\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2}{\|\omega\|_{L^2}^2} \mid \omega \neq 0 \perp_{L^2} \mathbf{H}^p(M, g) \right\}$$

$$\le \frac{\|d(f\varphi)\|_{L^2}^2 + \|d(f * \varphi)\|_{L^2}^2}{\|f\varphi\|_{L^2}^2} = \frac{\|df\|_{L^2}^2}{\|f\|_{L^2}^2} = \lambda_1^{(0)}(M, g).$$

From $\lambda_1^{(p)}(M, g) = \lambda_1^{(0)}(M, g)$, we have

$$\lambda_1^{(p)}(M, g) = \frac{\|d(f\varphi)\|_{L^2}^2 + \|\delta(f\varphi)\|_{L^2}^2}{\|f\varphi\|_{L^2}^2}.$$

Since $f\varphi$ attains the infimum, we see that $\Delta^{(p)}(f\varphi) = \lambda_1^{(p)}(M, g) f\varphi$, that is, $f\varphi$ is a first positive eigen p-form on (M, g). Thus, we have finished the proof of Theorem 1.1. □

We prove Proposition 1.2. The proof is the same as that of Theorem 1.1. In order to prove Proposition 1.2, in the proof of Theorem 1.1, it is enough to take a test space for the min–max principle as $E := \langle \varphi, f_1\varphi, \ldots, f_{b_p+k-1}\varphi \rangle_{\mathbb{R}}$, where φ is a non-trivial harmonic p-form of constant length and f_i is an ith eigenfunction on (M, g). Then, we have $\lambda_k^{(p)}(M, g) \le \lambda_{b_p+k-1}^{(0)}(M, g)$ for $k \ge 1$.

3. Examples

We give some examples for Theorem 1.1. First, we give an example such that the equality in Theorem 1.1 holds for $k = 1$.

Example 3.1. Let (T^m, g_0) be an m-dimensional flat torus. Then, all harmonic forms are parallel and $\lambda_1^{(p)}(T^m, g_0) = \lambda_1^{(0)}(T^m, g_0)$ for all p.

Next, we give an example such that the equality in Theorem 1.1 does not hold for $k = 1$.

Example 3.2. Let S^m be the m ($\geq$3)-dimensional sphere. By [3] when m is odd, and by [15,12] when m is even, there exists a family of metrics h_ε on S^m with the sectional curvature $K_{h_\varepsilon} \geq 0$ and $\mathrm{diam}(S^{2n}, h_\varepsilon) \leq d_1$ such that

$$\begin{aligned} &\lambda_1^{(0)}(S^m, h_\varepsilon) \geq C_1, \\ &\lambda_1^{(p)}(S^m, h_\varepsilon) \to 0 \text{ for } 1 \leq p \leq m-1, \end{aligned} \tag{3.1}$$

as $\varepsilon \to 0$, where d_1, C_1 are positive constants independent of ε.

We consider the $2m$-dimensional Riemannian manifold $(M, g_\varepsilon) := (S^m \times T^m, h_\varepsilon \oplus g_0)$. Since all harmonic p-forms on (M, g_ε) are parallel, by Theorem 1.1, we have $\lambda_1^{(p)}(M, g_\varepsilon) \leq \lambda_1^{(0)}(M, g_\varepsilon)$.

Since $\mathrm{Ric}_{g_\varepsilon} \geq 0$ and $\mathrm{diam}(M, g_\varepsilon) \leq d_2$, by [5,11], there exists a constant $C_2 > 0$ independent of ε such that

$$\lambda_1^{(0)}(M, g_\varepsilon) \geq C_2.$$

On the other hand, from the Künneth formula for the eigenvalues and (3.1), for $1 \leq p \leq m-1$ we have

$$\begin{aligned} \lambda_1^{(p)}(M, g_\varepsilon) &= \min_{a+b=p, i+j\geq 1}\{\lambda_i^{(a)}(S^m, h_\varepsilon) + \lambda_j^{(b)}(T^m, g_0)\} \leq \lambda_1^{(p)}(S^m, h_\varepsilon) \\ &\quad + \lambda_0^{(0)}(T^m, g_0) = \lambda_1^{(p)}(S^m, h_\varepsilon) \to 0 \text{ as } \varepsilon \to 0, \end{aligned}$$

where $\lambda_0^{(0)}$ means the zero eigenvalue. Therefore, for sufficiently small $\varepsilon > 0$, we have $\lambda_1^{(p)}(M, g_\varepsilon) < \lambda_1^{(0)}(M, g_\varepsilon)$ for $1 \leq p \leq m-1$.

The following example is based on Jammes [8], Example 1.2.

Example 3.3. Let H^3 be the 3-dimensional closed Heisenberg manifold, that is, $H = N/\Gamma$, where N is the 3-dimensional Heisenberg group

$$N := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

and Γ is the integer lattice in N. Then H is a 2-step nilmanifold and a T^2-bundle over S^1. Note that $b_1(H) = b_2(H) = 2$. The family of left invariant metrics h_ε on H is defined as in [8], Example 1.2, where $\alpha > 1$. All of non-trivial harmonic 1-forms and 2-forms on (H, h_ε) have constant length, while none of them are parallel.

We set $(M^4, g_\varepsilon) := (H^3 \times S^1, h_\varepsilon \oplus g_{S^1})$. Then all of non-trivial harmonic 2-forms on (M^4, g_ε) have the same property as above. By Theorem 1.1, we have $\lambda_1^{(2)}(M, g_\varepsilon) \leq \lambda_1^{(0)}(M, g_\varepsilon)$.

Since (M, g_ε) has bounded sectional curvature and bounded diameter, by [5,11], there exists a constant $C_3 > 0$ independent of ε such that

$$\lambda_1^{(0)}(M, g_\varepsilon) \geq C_3.$$

From Example 1.2 in [8] and the Künneth formula as in Example 3.2, we find that

$$\lambda_1^{(2)}(M, g_\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

Thus, for sufficiently small ε, the inequality $\lambda_1^{(2)}(M, g_\varepsilon) < \lambda_1^{(0)}(M, g_\varepsilon)$ holds.

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